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A numerical method for solving m-dimensional stochastic Itô–Volterra integral equations by stochastic operational matrix

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ABSTRACT

The multidimensional Itô–Volterra integral equations arise in many problems such as an exponential population growth model with several independent white noise sources. In this paper, we obtain a stochastic operational matrix of block pulse functions on interval [0, 1) to solve m-dimensional stochastic Itô–Volterra integral equations. By using block pulse functions and their stochastic operational matrix of integration, m-dimensional stochastic Itô–Volterra integral equations can be reduced to a linear lower triangular system which can be directly solved by forward substitution. We prove that the rate of convergence is $O(h)$. Furthermore, a 95% confidence interval of the errors' mean is made, the results shows that the approximate solutions have a credible degree of accuracy.

1. Introduction

We know that stochastic Itô–Volterra integral equations arise in many problems in mechanics, finance, biology, medical, social sciences, etc. So the study of such problems is very useful in application and there is an increasing demand for studying the behavior of a number of sophisticated dynamical systems in physical, medical and social sciences, as well as in engineering and finance. These systems are often dependent on a noise source, on a Gaussian white noise, for example, governed by certain probability laws, so that modeling such phenomena naturally requires the use of various stochastic differential equations [1–5] or, in more complicated cases, stochastic Itô–Volterra and Itô–Volterra–Fredholm integral equations and stochastic integro-differential equations [6–10]. Because in many problems such equations of course cannot be solved explicitly, it is important to find their approximate solutions by using some numerical methods [1–4,8–10].

In recent years, orthogonal functions or polynomials, such as block pulse functions, Walsh functions, Fourier series, Legendre polynomials, Chebyshev polynomials and Laguerre polynomials, were used to estimate solutions of some systems such as integral equations, [7–14]. In this paper we use of block pulse functions and stochastic integration operational matrix.

We consider the following m-dimensional linear stochastic Itô–Volterra integral equation,

$$X(t) = f(t) + \int_0^t \mu(s,t)X(s)ds + \sum_{j=1}^{m} \int_0^t \sigma_j(s,t)X(s)dB_j(s) \quad t \in [0,T),$$

where $X(t), f(t)$, $\mu(s,t)$ and $\sigma_j(s,t)$, $j = 1,2,\ldots,m$, for $s,t \in [0,T)$, are the stochastic processes defined on the same probability space (Ω, F, P), and $X(t)$ is unknown. Also $B(t) = (B_1(t), B_2(t), \ldots, B_m(t))$ is an m-dimensional Brownian motion process and $\int_0^t \sigma_j(s,t)X(s)dB_j(s)$, $j = 1,2,\ldots,m$, are the Itô integrals.

This paper is organized as follows.

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In Section 2, we describe the basic properties of the block pulse functions and functions approximation by block pulse functions and integration operational matrix. In Section 3, we obtain the stochastic integration operational matrix. In Section 4, we solve stochastic Itô–Volterra integral equations with several independent white noise sources by using a stochastic integration operational matrix. Section 5 is devoted to error analysis proposed method and in Section 6, we report our numerical finding and demonstrate the accuracy of the proposed scheme by considering numerical examples. Finally, Section 7 gives some brief conclusions.

2. Block pulse functions (BPFs)

BPFs have been studied by many authors and applied for solving different problems; for example, see [7–14]. The goal of this section is to recall notations and definition of the block pulse functions, state some known results, and derive useful formulas that are important for this paper. These have discussed thoroughly in [15,16].

2.1. Definition of BPFs

We define the m-set of BPFs as

$$\phi_i(t) = \begin{cases} 1 & (i-1)h \le t < ih, \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

with $t \in [0, T),\ i = 1, 2, \ldots, m$ and $h = \frac{T}{m}$.

The elementary properties of BPFs are as follows

(1) *Disjointness*: The BPFs are disjointed with each other in the interval $t \in [0, T)$

$$\phi_i(t)\phi_j(t) = \delta_{ij}\phi_i(t), \tag{2}$$

where $i, j = 1, 2, \ldots, m$ and δ_{ij} is Kronecker delta.

(2) *Orthogonality*: The BPFs are orthogonal with each other in the interval $t \in [0, T)$

$$\int_0^T \phi_i(t)\phi_j(t)dt = h\delta_{ij}, \quad i, j = 1, 2, \ldots, m. \tag{3}$$

(3) *Completeness*: If $m \to \infty$, then the BPFs set is complete; i.e. for every $f \in L^2([0, T))$, Parseval's identity holds,

$$\int_0^T f^2(t)dt = \sum_{i=1}^{\infty} f_i^2 \|\phi_i(t)\|^2, \tag{4}$$

where

$$f_i = \frac{1}{h}\int_0^T f(t)\phi_i(t)dt. \tag{5}$$

Vector form: Consider the first m terms of BPFs and write them concisely as m-vector

$$\Phi(t) = \left(\phi_1(t), \phi_2(t), \ldots, \phi_m(t)\right)^T, \quad t \in [0, T).$$

The above representation and disjointness property follows

$$\Phi(t)\Phi^T(t) = \begin{pmatrix} \phi_1(t) & 0 & \cdots & 0 \\ 0 & \phi_2(t) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \phi_m(t) \end{pmatrix}_{m\times m}, \tag{6}$$

furthermore, we have

$$\Phi^T(t)\Phi(t) = 1,$$

and,

$$\Phi(t)\Phi^T(t)F^T = D_F\Phi(t), \tag{7}$$

where D_F usually denotes a diagonal matrix whose diagonal entries are related to a constant vector $F = \left(f_1, f_2, \ldots, f_m\right)^T$.

2.2. Functions approximation

An arbitrary real bounded function $f(t)$, which is square integrable in the interval $t \in [0, T)$, can be expanded into a block pulse series in the sense of minimizing the mean square error between $f(t)$ and its approximation

$$f(t) \simeq \hat{f}_m(t) = \sum_{i=1}^{m} f_i \phi_i(t), \tag{8}$$

where f_i is the block pulse coefficient with respect to the ith BPF $\phi_i(t)$. In the vector form we have,

$$f(t) \simeq \hat{f}_m(t) = F^T \Phi(t) = \Phi^T(t) F, \tag{9}$$

where

$$F = \left(f_1, f_2, \ldots, f_m\right)^T.$$

Let $k(s,t) \in L^2\big([0, T_1) \times [0, T_2)\big)$. It can be similarly expanded with respect to BPFs such as

$$k(s,t) = \Psi^T(s) K \Phi(t) = \Phi^T(t) K^T \Psi(s), \tag{10}$$

where $\Psi(s)$ and $\Phi(t)$ are m_1 and m_2 dimensional BPFs vectors respectively, and $K = \left(k_{ij}\right)$, $i = 1, 2, \ldots, m_1$, $j = 1, 2, \ldots, m_2$ is the $m_1 \times m_2$ block pulse coefficient matrix with

$$k_{ij} = \frac{1}{h_1 h_2} \int_0^{T_1} \int_0^{T_2} k(s,t) \Psi_i(s) \Phi_j(t) dt ds,$$

where $h_1 = \frac{T_1}{m_1}$, $h_2 = \frac{T_2}{m_2}$. For convenience, we put $m_1 = m_2 = m$.

2.3. Integration operational matrix

Computing $\int_0^t \phi_i(s)ds$ follows

$$\int_0^t \phi_i(s)ds = \begin{cases} 0 & 0 \le t < (i-1)h, \\ t-(i-1)h & (i-1)h \le t < ih, \\ h & ih \le t < T. \end{cases} \tag{11}$$

Since $t-(i-1)h$, equals to $\frac{h}{2}$, at the mid-point of $[(i-1)h, ih)$, we can approximate $t-(i-1)h$, for $(i-1)h \le t < ih$, by $\frac{h}{2}$.

Now expressing $\int_0^t \phi_i(s)ds$, in terms of the BPFs follows

$$\int_0^t \phi_i(s)ds \simeq \left(0, \ldots, 0, \frac{h}{2}, h, \ldots, h\right) \Phi(t), \tag{12}$$

where $\frac{h}{2}$ is the ith component of vector.

Therefore, [15],

$$\int_0^t \Phi(s)ds \simeq P\Phi(t), \tag{13}$$

where the operational matrix of integration is given by

$$P = \frac{h}{2} \begin{pmatrix} 1 & 2 & 2 & \cdots & 2 \\ 0 & 1 & 2 & \cdots & 2 \\ 0 & 0 & 1 & \cdots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}_{m \times m}. \tag{14}$$

So, the integral of every function $f(t)$ can be approximated as follows

$$\int_0^t f(s)ds \simeq \int_0^t F^T \Phi(s)ds \simeq F^T P \Phi(t). \tag{15}$$

3. Stochastic integration operational matrix

The Itô integral of each single BPFs $\phi_i(t)$ can be computed as follows,

$$\int_0^t \phi_i(s)dB(s) = \begin{cases} 0 & 0 \le t < (i-1)h, \\ B(t) - B((i-1)h) & (i-1)h \le t < ih, \\ B(ih) - B((i-1)h) & ih \le t < T. \end{cases} \tag{16}$$

Since $B(t) - B((i-1)h)$ is equal to $B((i-0.5)h) - B((i-1)h)$, at the mid-point of $[(i-1)h, ih)$, we can approximate $B(t) - B((i-1)h)$, for $(i-1)h \le t < ih$, by $B((i-0.5)h) - B((i-1)h)$.

Now expressing $\int_0^t \phi_i(s)dB(s)$, in terms of the BPFs follows

$$\int_0^t \phi_i(s)dB(s) \simeq \Big(B((i-0.5)h) - B((i-1)h)\Big)\phi_i(t) + \Big(B(ih) - B((i-1)h)\Big) \sum_{j=i+1}^{m} \phi_j(t), \tag{17}$$

and it has the vector form,

$$\int_0^t \phi_i(s)dB(s) \simeq \Big(0, \ldots, 0, B((i-0.5)h) - B((i-1)h), B(ih) - B((i-1)h), \ldots, B(ih) - B((i-1)h)\Big)\Phi(t), \tag{18}$$

in which the ith component is $B((i-0.5)h) - B((i-1)h)$.

Therefore

$$\int_0^t \Phi(s)dB(s) \simeq P_S \Phi(t), \tag{19}$$

where the stochastic operational matrix of integration is given by

$$P_S = \begin{pmatrix} B\left(\frac{h}{2}\right) & B(h) & B(h) & \cdots & B(h) \\ 0 & B\left(\frac{3h}{2}\right) - B(h) & B(2h) - B(h) & \cdots & B(2h) - B(h) \\ 0 & 0 & B\left(\frac{5h}{2}\right) - B(2h) & \cdots & B(3h) - B(2h) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & B\left(\frac{(2m-1)h}{2}\right) - B((m-1)h) \end{pmatrix}_{m \times m}. \tag{20}$$

So, the Itô integral of every function $f(t)$ can be approximated as follows

$$\int_0^t f(s)dB(s) \simeq \int_0^t F^T \Phi(s)dB(s) \simeq F^T P_S \Phi(t). \tag{21}$$

4. Solving *m*-dimensional stochastic Itô–Volterra integral equations by stochastic operational matrix

We consider following linear stochastic Itô–Volterra integral equation with several independent white noise sources,

$$X(t) = f(t) + \int_0^t \mu(s,t)X(s)ds + \sum_{j=1}^{m} \int_0^t \sigma_j(s,t)X(s)dB_j(s) \quad t \in [0, T). \tag{22}$$

Our problem is to determine the block pulse coefficient of $X(t)$, where $X(t), f(t), \mu(s,t)$ and $\sigma_j(s,t)$, $j = 1, 2, \ldots, m$, for $s, t \in [0, T)$, are the stochastic processes defined on the same probability space $(\Omega, \mathcal{F}, P)$. Also $B(t) = (B_1(t), B_2(t), \ldots, B_m(t))$ is an m-dimensional Brownian motion process and $\int_0^t \sigma_j(s,t)X(s)dB_j(s)$, $j = 1, 2, \ldots, m$, are the Itô integrals.

We approximate $X(t), f(t), \mu(s,t)$ and $\sigma_j(s,t)$, $j = 1, 2, \ldots, m$, by relations (9), (10) as follows

$$\begin{aligned} &X(t) \simeq X^T \Phi(t) = \Phi^T(t) X, \\ &f(t) \simeq F^T \Phi(t) = \Phi^T(t) F, \\ &\mu(s,t) \simeq \Psi^T(s) \Lambda \Phi(t) = \Phi^T(t) \Lambda^T \Psi(s), \\ &\sigma_j(s,t) \simeq \Psi^T(s) \Sigma_j \Phi(t) = \Phi^T(t) \Sigma_j^T \Psi(s), \quad j = 1, 2, \ldots, m. \end{aligned}$$

In the above approximates, X and F are stochastic block pulse coefficient vectors, and Λ and Σ_j, $j = 1, 2, \dots, m$, are stochastic block pulse coefficient matrices.

With substituting above approximation in Eq. (22), we get

$$X^T\Phi(t) \simeq F^T\Phi(t) + X^T\left(\int_0^t \Psi(s)\Psi^T(s)ds\right)\Lambda\Phi(t) + X^T\left(\sum_{j=1}^m\left(\int_0^t \Psi(s)\Psi^T(s)dB_j(s)\right)\Sigma_j\right)\Phi(t). \tag{23}$$

Let Λ^i and Σ_j^i, $j = 1, 2, \dots, m$ be the ith row of the constant matrices Λ and Σ_j, for $j = 1, 2, \dots, m$; R^i be the ith row of the integration operational matrix P; R_S^i be the ith row of the stochastic integration operational matrix P_S; D_{Λ^i} and $D_{\Sigma_j^i}$ are diagonal matrices with Λ^i and Σ_j^i, $j = 1, 2, \dots, m$, as its diagonal entries. By the previous relations and assuming $m_1 = m_2$, we have,

$$\begin{aligned}\left(\int_0^t \Psi(s)\Psi^T(s)ds\right)\Lambda\Phi(t) &= \left(\int_0^t \Phi(s)\Phi^T(s)ds\right)\Lambda\Phi(t)\\ &= \begin{pmatrix} R^1\Phi(t)\Lambda^1\Phi(t)\\ R^2\Phi(t)\Lambda^2\Phi(t)\\ \vdots\\ R^m\Phi(t)\Lambda^m\Phi(t)\end{pmatrix} = \begin{pmatrix} R^1D_{\Lambda^1}\\ R^2D_{\Lambda^2}\\ \vdots\\ R^mD_{\Lambda^m}\end{pmatrix}\Phi(t) = A\Phi(t),\end{aligned} \tag{24}$$

where

$$A = \frac{h}{2}\begin{pmatrix} \lambda_{11} & 2\lambda_{12} & 2\lambda_{13} & \cdots & 2\lambda_{1m}\\ 0 & \lambda_{22} & 2\lambda_{23} & \cdots & 2\lambda_{2m}\\ 0 & 0 & \lambda_{33} & \cdots & 2\lambda_{3m}\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & \lambda_{mm}\end{pmatrix}_{m\times m}, \tag{25}$$

and also for the Itô integral terms, we have

$$\begin{aligned}\left(\int_0^t \Psi(s)\Psi^T(s)dB(s)\right)\Sigma_j\Phi(t) &= \left(\int_0^t \Phi(s)\Phi^T(s)dB(s)\right)\Sigma_j\Phi(t)\\ &= \begin{pmatrix} R_S^1\Phi(t)\Sigma_j^1\Phi(t)\\ R_S^2\Phi(t)\Sigma_j^2\Phi(t)\\ \vdots\\ R_S^m\Phi(t)\Sigma_j^m\Phi(t)\end{pmatrix} = \begin{pmatrix} R_S^1D_{\Sigma_j^1}\\ R_S^2D_{\Sigma_j^2}\\ \vdots\\ R_S^mD_{\Sigma_j^m}\end{pmatrix}\Phi(t) = A_j\Phi(t),\end{aligned} \tag{26}$$

where

$$A_j = \begin{pmatrix} \sigma_{11}^jB\left(\frac{h}{2}\right) & \sigma_{12}^jB(h) & \sigma_{13}^jB(h) & \cdots & \sigma_{1m}^jB(h)\\ 0 & \sigma_{22}^j\left(B\left(\frac{3h}{2}\right) - B(h)\right) & \sigma_{23}^j\left(B(2h) - B(h)\right) & \cdots & \sigma_{2m}^j\left(B(2h) - B(h)\right)\\ 0 & 0 & \sigma_{33}^j\left(B\left(\frac{5h}{2}\right) - B(2h)\right) & \cdots & \sigma_{3m}^j\left(B(3h) - B(2h)\right)\\ \vdots & \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & \cdots & \sigma_{mm}^j\left(B\left(\frac{(2m-1)h}{2}\right) - B((m-1)h)\right)\end{pmatrix}. \tag{27}$$

With substituting relations (24) and (26) in (23), we get

$$X^T\Phi(t) \simeq F^T\Phi(t) + X^TA\Phi(t) + X^T\left(\sum_{j=1}^m A_j\right)\Phi(t).$$

Then,

$$X^T\left(I - A - \sum_{j=1}^m A_j\right) \simeq F^T. \tag{28}$$

So, by setting $M = \left(I - A - \sum_{j=1}^{m} A_j\right)^T$ and replacing $\simeq$ by $=$, we will have,

$$MX = F. \tag{29}$$

Which is a linear system of equations with lower triangular coefficients matrix that gives the approximate block pulse coefficient of the unknown stochastic processes $X(t)$.

5. Error analysis

In this section, we will show that the rate of convergence presented method for solving stochastic Itô–Volterra integral equations with several independent white noise sources is $O(h)$.

Theorem 1. *Suppose that $f(t)$ is an arbitrary real bounded function, which is square integrable in the interval $[0, 1)$, and $e(t) = f(t) - \hat{f}_m(t),\ t \in I = [0, 1)$, which $\hat{f}_m(t) = \sum_{i=1}^{m} f_i\phi_i(t)$ is the block pulse series of $f(t)$. Then,*

$$\|e(t)\| \le \frac{h}{2\sqrt{3}} \sup_{t\in I} |f'(t)|. \tag{30}$$

Proof. Let,

$$e_i(t) = \begin{cases} f(t) - f_i & t \in D_i, \\ 0 & t \in I - D_i \end{cases} \tag{31}$$

where $D_i = \{t : (i-1)h \le t < ih,\ h = \frac{1}{m}\}$ and $i = 1, 2, \dots, m$.

We have,

$$e_i(t) = f(t) - \frac{1}{h}\int_{(i-1)h}^{ih} f(s)ds = \frac{1}{h}\int_{(i-1)h}^{ih} \Big(f(t) - f(s)\Big)ds,$$

now by mean value theorem, we get,

$$e_i(t) = \frac{f'(\eta_i)}{h}\int_{(i-1)h}^{ih} (t-s)ds = f'(\eta_i)\left(t + \left(-i + \frac{1}{2}\right)h\right), \quad t, \eta_i \in D_i,\ i = 1, 2, \dots, m$$

then,

$$\begin{aligned} \|e_i(t)\|^2 &= \int_{(i-1)h}^{ih} |e_i(t)|^2 dt = \big(f'(\eta_i)\big)^2 \int_{(i-1)h}^{ih} \left(t + \left(-i + \frac{1}{2}\right)h\right)^2 dt \\ &= \frac{h^3}{12}\big(f'(\eta_i)\big)^2, \quad \eta_i \in D_i,\ i = 1, 2, \dots, m. \end{aligned} \tag{32}$$

Consequently

$$\begin{aligned} \|e(t)\|^2 &= \int_0^1 |e(t)|^2 dt = \int_0^1 \left(\sum_{i=1}^{m} e_i(t)\right)^2 dt \\ &= \int_0^1 \left[\sum_{i=1}^{m} e_i^2(t) + 2\sum_{i<j} e_i(t)e_j(t)\right]dt = \sum_{i=1}^{m}\int_0^1 e_i^2(t)dt = \sum_{i=1}^{m} \|e_i(t)\|^2 \\ &= \frac{h^3}{12}\sum_{i=1}^{m}\big(f'(\eta_i)\big)^2 \le \frac{h^2}{12}\sup_{t\in I}|f'(t)|^2, \end{aligned} \tag{33}$$

or,

$$\|e(t)\| \le \frac{h}{2\sqrt{3}} \sup_{t\in I} |f'(t)|$$

hence, $\|e(t)\| = O(h)$. □

Theorem 2. *Suppose that $f(s, t) \in L^2\big([0, 1) \times [0, 1)\big)$ and $e(s, t) = f(s, t) - \hat{f}_m(s, t)$, $(s, t) \in D = [0, 1) \times [0, 1)$, which $\hat{f}_m(s, t) = \sum_{i=1}^{m}\sum_{j=1}^{m} f_{ij}\psi_i(s)\phi_j(t)$ is the block pulse series of $f(s, t)$. Then,*

$$\|e(s, t)\| \le \frac{h}{2\sqrt{3}}\left(\sup_{(x,y)\in D}|f_s'(x, y)|^2 + \sup_{(x,y)\in D}|f_t'(x, y)|^2\right)^{\frac{1}{2}}. \tag{34}$$

Proof. Let,

$$e_{ij}(s,t)=\begin{cases} f(s,t)-f_{ij} & (s,t)\in D_{ij},\\ 0 & (s,t)\in D-D_{ij}\end{cases} \tag{35}$$

where $D_{ij}=\{(s,t):(i-1)h\le s<ih,\ (j-1)h\le t<jh,\ h=\frac{1}{m}\}$ and $i,j=1,2,\dots,m$.

For $i,j=1,2,\dots,m$, we have,

$$e_{ij}(s,t)=f(s,t)-\frac{1}{h^2}\int_{(i-1)h}^{ih}\int_{(j-1)h}^{jh} f(x,y)dydx=\frac{1}{h^2}\int_{(i-1)h}^{ih}\int_{(j-1)h}^{jh}\Big(f(s,t)-f(x,y)\Big)dydx,$$

now by mean value theorem, we get,

$$\begin{aligned} e_{ij}(s,t) &= \frac{1}{h^2}\int_{(i-1)h}^{ih}\int_{(j-1)h}^{jh}\Big((s-x)f_s'(\eta_i,\eta_j)+(t-y)f_t'(\eta_i,\eta_j)\Big)dydx\\ &= f_s'(\eta_i,\eta_j)\left(s+\left(-i+\frac{1}{2}\right)h\right)+f_t'(\eta_i,\eta_j)\left(t+\left(-j+\frac{1}{2}\right)h\right),\quad (s,t),(\eta_i,\eta_j)\in D_{ij}\end{aligned}$$

then,

$$\begin{aligned} \|e_{ij}(s,t)\|^2 &= \int_{(i-1)h}^{ih}\int_{(j-1)h}^{jh}|e_{ij}(s,t)|^2dtds\\ &= \frac{h^4}{12}\Big(f_s'^2(\eta_i,\eta_j)+f_t'^2(\eta_i,\eta_j)\Big),\quad (\eta_i,\eta_j)\in D_{ij},\ i,j=1,2,\dots,m. \end{aligned} \tag{36}$$

Consequently

$$\begin{aligned} \|e(s,t)\|^2 &= \int_0^1\int_0^1|e(s,t)|^2dtds=\int_0^1\int_0^1\left(\sum_{i=1}^m\sum_{j=1}^m e_{ij}(s,t)\right)^2dtds\\ &= \sum_{i=1}^m\sum_{j=1}^m\int_0^1\int_0^1 e_{ij}^2(s,t)dtds=\sum_{i=1}^m\sum_{j=1}^m\|e_{ij}(s,t)\|^2\\ &= \frac{h^4}{12}\sum_{i=1}^m\sum_{j=1}^m\Big(f_s'^2(\eta_i,\eta_j)+f_t'^2(\eta_i,\eta_j)\Big)\le\frac{h^2}{12}\Big(\sup_{(x,y)\in D}|f_s'(x,y)|^2+\sup_{(x,y)\in D}|f_t'(x,y)|^2\Big), \end{aligned} \tag{37}$$

or,

$$\|e(s,t)\|\le\frac{h}{2\sqrt{3}}\Big(\sup_{(x,y)\in D}|f_s'(x,y)|^2+\sup_{(x,y)\in D}|f_t'(x,y)|^2\Big)^{\frac{1}{2}}$$

hence, $\|e(s,t)\|=O(h)$. □

Theorem 3. *Suppose $X(t)$ is the exact solution of (22) and $\hat{X}_m(t)$ is the block pulse series approximate solution of (22) that their coefficients are obtained by (29). Also assume that*

(1) $\|X(t)\|\le\rho,\ t\in I=[0,1)$,
(2) $\|\mu(s,t)\|\le M,\ (s,t)\in D=I\times I$,
(3) $\|\sigma_j(s,t)\|\le M_j,\ j=1,2,\dots,m\ (s,t)\in D=I\times I$,
(4) $M+\gamma(h)+\sum_{j=1}^m\Big(M_j+\gamma_j(h)\Big)\sup_{t\in I}|B_j(t)|<1$,

then

$$\|X(t)-\hat{X}_m(t)\|\le\frac{\Gamma(h)+\gamma(h)\rho+\rho\sum_{j=1}^m\gamma_j(h)\sup_{t\in I}|B_j(t)|}{1-\left(M+\gamma(h)+\sum_{j=1}^m\big(M_j+\gamma_j(h)\big)\sup_{t\in I}|B_j(t)|\right)},\quad t\in I \tag{38}$$

where

$$\|X(t)\|=\Big(E\big[X^2(t)\big]\Big)^{\frac{1}{2}},$$

$$\Gamma(h) = \frac{h}{2\sqrt{3}} \sup_{t\in I} |f'(t)|,$$
$$\gamma(h) = \frac{h}{2\sqrt{3}}\Big(\sup_{(x,y)\in D} |\mu'_s(x,y)|^2 + \sup_{(x,y)\in D} |\mu'_t(x,y)|^2\Big)^{\frac{1}{2}},$$
$$\gamma_j(h) = \frac{h}{2\sqrt{3}}\Big(\sup_{(x,y)\in D} |\sigma'_{js}(x,y)|^2 + \sup_{(x,y)\in D} |\sigma'_{jt}(x,y)|^2\Big)^{\frac{1}{2}}, \quad j = 1,2,\ldots,m.$$

Proof. By using Theorems 1 and 2, we have,

$$\|f(t) - \hat{f}_m(t)\| \le \frac{h}{2\sqrt{3}} \sup_{t\in I} |f'(t)| = \Gamma(h), \tag{39}$$

and

$$\|\mu(s,t) - \hat{\mu}_m(s,t)\| \le \frac{h}{2\sqrt{3}}\Big(\sup_{(x,y)\in D} |\mu'_s(x,y)|^2 + \sup_{(x,y)\in D} |\mu'_t(x,y)|^2\Big)^{\frac{1}{2}} = \gamma(h), \tag{40}$$

and

$$\|\sigma_j(s,t) - \hat{\sigma}_{jm}(s,t)\| \le \frac{h}{2\sqrt{3}}\Big(\sup_{(x,y)\in D} |\sigma'_{js}(x,y)|^2 + \sup_{(x,y)\in D} |\sigma'_{jt}(x,y)|^2\Big)^{\frac{1}{2}} = \gamma_j(h), \quad j = 1,2,\ldots,m. \tag{41}$$

From (22), we get

$$\begin{aligned} X(t) - \hat{X}_m(t) &= f(t) - \hat{f}_m(t) + \int_0^t \Big(\mu(s,t)X(s) - \hat{\mu}_m(s,t)\hat{X}_m(s)\Big)ds \\ &\quad + \sum_{j=1}^m \int_0^t \Big(\sigma_j(s,t)X(s) - \hat{\sigma}_{jm}(s,t)\hat{X}_m(s)\Big)dB_j(s), \end{aligned} \tag{42}$$

then by mean value theorem, we can write

$$\begin{aligned} \|X(t) - \hat{X}_m(t)\| &\le \|f(t) - \hat{f}_m(t)\| + t\|\mu(s,t)X(s) - \hat{\mu}_m(s,t)\hat{X}_m(s)\| \\ &\quad + \sum_{j=1}^m B_j(t)\|\sigma_j(s,t)X(s) - \hat{\sigma}_{jm}(s,t)\hat{X}_m(s)\|. \end{aligned} \tag{43}$$

By using (H1), (H2) and (40), we have

$$\begin{aligned} \|\mu(s,t)X(s) - \hat{\mu}_m(s,t)\hat{X}_m(s)\| &\le \|\mu(s,t)\|\,\|X(s) - \hat{X}_m(s)\| + \|\mu(s,t) - \hat{\mu}_m(s,t)\|\Big(\|X(s) - \hat{X}_m(s)\| + \|X(s)\|\Big) \\ &\le (M + \gamma(h))\|X(s) - \hat{X}_m(s)\| + \gamma(h)\rho, \end{aligned} \tag{44}$$

and

$$\begin{aligned} &\|\sigma_j(s,t)X(s) - \hat{\sigma}_{jm}(s,t)\hat{X}_m(s)\| \\ &\quad \le \|\sigma_j(s,t)\|\,\|X(s) - \hat{X}_m(s)\| + \|\sigma_j(s,t) - \hat{\sigma}_{jm}(s,t)\|\Big(\|X(s) - \hat{X}_m(s)\| + \|X(s)\|\Big) \\ &\quad \le (M_j + \gamma_j(h))\|X(s) - \hat{X}_m(s)\| + \gamma_j(h)\rho, \quad j = 1,2,\ldots,m. \end{aligned} \tag{45}$$

So

$$\begin{aligned} \|X(t) - \hat{X}_m(t)\| &\le \Gamma(h) + t\Big(\Big(M + \gamma(h)\Big)\|X(t) - \hat{X}_m(t)\| + \gamma(h)\rho\Big) \\ &\quad + \sum_{j=1}^m B_j(t)\Big(\Big(M_j + \gamma_j(h)\Big)\|X(t) - \hat{X}_m(t)\| + \gamma_j(h)\rho\Big), \end{aligned} \tag{46}$$

or

$$\|X(t) - \hat{X}_m(t)\| \le \frac{\Gamma(h) + \gamma(h)\rho + \rho\sum_{j=1}^m \gamma_j(h) \sup_{t\in I}|B_j(t)|}{1 - \left(M + \gamma(h) + \sum_{j=1}^m \Big(M_j + \gamma_j(h)\Big)\sup_{t\in I}|B_j(t)|\right)}, \quad t \in I \tag{47}$$

hence, $\|X(t) - \hat{X}_m(t)\| = O(h)$. □

Table 1
Mean, standard deviation and mean confidence interval for error in Example 1 with $n = 20$.

m	$\bar{x}_E$	s_E	0.95 confidence interval for mean of E	
			Lower	*Upper*
4	2.80383739E−4	1.42692696E−4	2.17845908E−4	3.42921571E−4
8	4.89201948E−4	1.42851647E−4	4.26594454E−4	5.51809443E−4
16	6.52973571E−4	1.10050413E−4	6.04741850E−4	7.01205291E−4
32	0.00121313404	2.14221469E−4	0.00111924509	0.00130701849
48	0.00172837457	1.92647371E−4	0.00164394314	0.00181280600
64	0.00218103587	2.38125316E−4	0.00207667285	0.00228539889
80	0.00254096373	3.96350688E−4	0.00236725539	0.00271467208

Table 2
Mean, standard deviation and mean confidence interval for error in Example 1 with $n = 20$.

m	$\bar{x}_E$	s_E	0.95 confidence interval for mean of E	
			Lower	*Upper*
4	2.80383739E−4	1.42692696E−4	2.17845908E−4	3.42921571E−4
8	4.89201948E−4	1.42851647E−4	4.26594454E−4	5.51809443E−4
16	6.52973571E−4	1.10050413E−4	6.04741850E−4	7.01205291E−4
32	0.00121313404	2.14221469E−4	0.00111924509	0.00130701849
48	0.00172837457	1.92647371E−4	0.00164394314	0.00181280600
64	0.00218103587	2.38125316E−4	0.00207667285	0.00228539889
80	0.00254096373	3.96350688E−4	0.00236725539	0.00271467208

6. Numerical examples

To illustrate the method stated in Section 4, we consider below some examples. The computations associated with the examples were performed using Matlab 7. Let X_i denote the block pulse coefficient of exact solution of the given examples, and let Y_i be the block pulse coefficient of computed solutions by the presented method. The error is defined as

$$\|E\|_\infty = \max_{1\le i\le m} |X_i - Y_i|.$$

Example 1. Consider the following linear stochastic Itô–Volterra integral equation,

$$X(t) = X_0 + \int_0^t rX(s)ds + \sum_{j=1}^{m} \int_0^t \alpha_j X(s)dB_j(s) \quad s, t \in [0, 1), \tag{48}$$

with the exact solution $X(t) = X_0 e^{(r-\frac{1}{2}\sum_{j=1}^m \alpha_j^2)t+\sum_{j=1}^m \alpha_j B_j(t)}$, for $0 \le t < 1$, where $X(t)$ is an unknown stochastic processes defined on the probability space $(\Omega, \mathcal{F}, P)$, and $B(t) = (B_1(t), B_2(t), \dots, B_m(t))$ is an m-dimensional Brownian motion process. The numerical results for $X_0 = \frac{1}{200}, r = \frac{1}{20}, \alpha_1 = \frac{1}{50}, \alpha_2 = \frac{2}{50}, \alpha_3 = \frac{4}{50}$ and $\alpha_4 = \frac{9}{50}$ are shown in Table 1. In Table 1, n is the number of iterations, $\bar{x}_E$ is the mean of error, and s_E is the standard deviation of error. The curves in Fig. 1 represent a trajectory of the approximate solution computed by the presented method with a trajectory of the exact solution. The curves in Fig. 2 represent the variation process of error.

Example 2. Consider the following linear stochastic Itô–Volterra integral equation,

$$X(t) = X_0 + \int_0^t r(s)X(s)ds + \sum_{j=1}^{m} \int_0^t \alpha_j(s)X(s)dB_j(s) \quad s, t \in [0, 1), \tag{49}$$

with the exact solution $X(t) = X_0 e^{\int_0^t (r(s)-\frac{1}{2}\sum_{j=1}^m \alpha_j^2(s))ds+\sum_{j=1}^m \int_0^t \alpha_j(s)dB_j(s)}$, for $0 \le t < 1$, where $X(t)$ is an unknown stochastic process defined on the probability space $(\Omega, \mathcal{F}, P)$, and $B(t) = (B_1(t), B_2(t), \dots, B_m(t))$ is an m-dimensional Brownian motion process. The numerical results for $X_0 = \frac{1}{12}, r(s) = s^2, \alpha_1(s) = \sin(s), \alpha_2(s) = \cos(s)$ and $\alpha_3(s) = s$ are shown in Table 2. In Table 2, n is the number of iterations, $\bar{x}_E$ is mean of error, and s_E is the standard deviation of error. The curves in Fig. 3 represent a trajectory of the approximate solution computed by the presented method with a trajectory of exact solution. The curves in Fig. 4 represent the variation process of error.

7. Conclusion

Because it is almost impossible to find the exact solution of Eq. (22), it would be convenient to determine its numerical solution based on stochastic numerical analysis. Using block pulse functions as basis functions to solve the linear stochastic

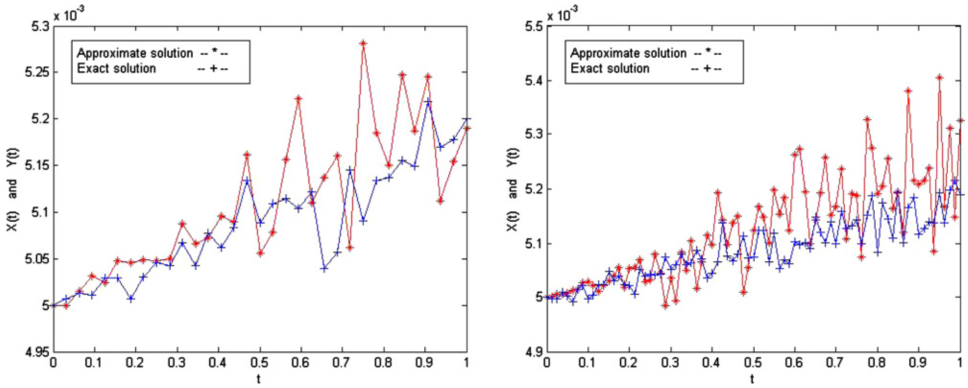

Fig. 1. The trajectory of the approximate solution and exact solution of Example 1 for $m = 32$, $m = 80$, $n = 20$.

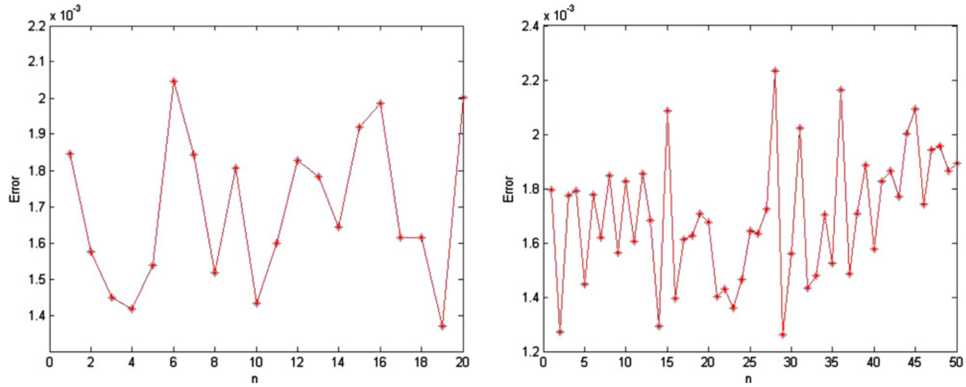

Fig. 2. Variation trend of error in Example 1 for $m = 48$, $n = 20$, $n = 50$.

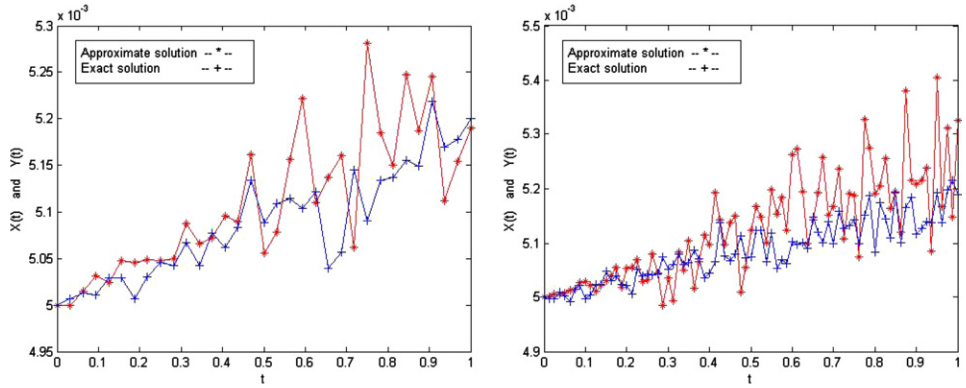

Fig. 3. The trajectory of the approximate solution and exact solution of Example 1 for $m = 32$, $m = 80$, $n = 20$.

Itô–Volterra integral equations with several independent white noise sources is very simple and effective in comparison with other methods. Its applicability and accuracy is checked on some examples.

The advantages of using the block pulse functions are simple calculations and conversion of the integral equation to a triangle system. So, by using block pulse functions and their stochastic operational matrix for stochastic Itô–Volterra integral equation, oscillations appear. Fluctuations generated in the exact and approximated answers are not dependant on a selected basis, but on random factors in the equation. Oscillations that appeared in the answers are due to Itô integrals in the integral equation. Itô integral is related to the Brownian motion. Simulation of Brownian motion $B(t)$, is related to random numbers with normal distribution, since random number have fluctuations, these fluctuations are transmitted to the responses. With this method, different random paths for simulated Brownian motion is obtained, and we therefore consider the average of the directions. We used the article [17] for the simulation of Brownian motion.

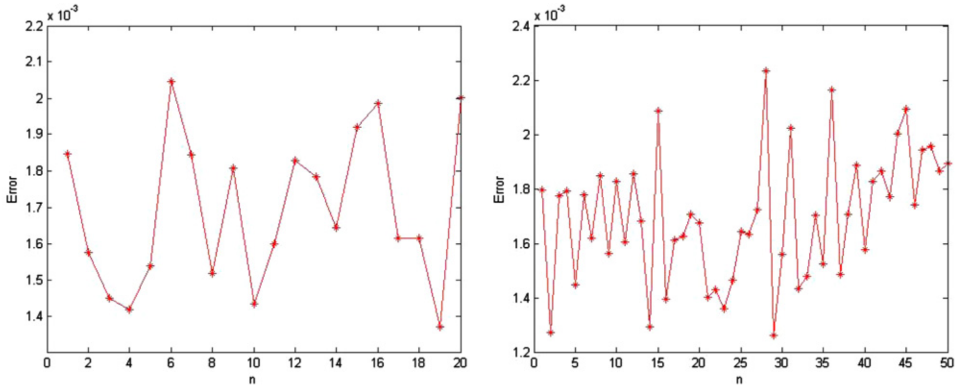

Fig. 4. Variation trend of error in Example 1 for $m = 48$, $n = 20$, $n = 50$.

Moreover, one could also apply the Itô–Taylor expansion described by Kloeden and Platen [3], or those from article [18], for example. Certainly, it could be the topic of some future work.

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